

[2010]47B32, 47B35

# GENERALIZED BUNDLE SHIFT WITH APPLICATION TO MULTIPLICATION OPERATOR ON THE BERGMAN SPACE

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**ABSTRACT.** Following upon results of Putinar, Sun, Wang, Zheng and the first author, we provide models for the restrictions of the multiplication by a finite Blaschke product on the Bergman space in the unit disc to its reducing subspaces. The models involve a generalization of the notion of bundle shift on the Hardy space introduced by Abrahamse and the first author to the Bergman space. We develop generalized bundle shifts on more general domains. While the characterization of the bundle shift is rather explicit, we have not been able to obtain all the earlier results appeared, in particular, the facts that the number of the minimal reducing subspaces equals the number of connected components of the Riemann surface  $B(z) = B(w)$  and the algebra of commutant of  $T_B$  is commutative, are not proved. Moreover, the role of the Riemann surface is not made clear also.

Bergman Bundle Shift, Bergman space, Finite Blaschke product, Reducing subspace

## 1. INTRODUCTION

In the study of bounded linear operators, we can ask about the reducing subspaces of a general bounded operator defined on a separable Hilbert space. But, in general, we can not say much about reducing subspace of arbitrary bounded linear operators. If we restrict attention to some special class of operators, then we can get more information about their reducing subspaces. One set of examples consist of multiplication operators by finite Blaschke products on the Bergman space on the unit disc  $\mathbb{D}$ .

Let  $\mathcal{O}(\mathbb{D})$  be the set of holomorphic functions on  $\mathbb{D}$  and  $L_a^2(\mathbb{D})$  be the Bergman space of functions in  $\mathcal{O}(\mathbb{D})$  satisfying

$$\|f\|^2 = -\frac{1}{2\pi i} \int_{\mathbb{D}} |f(z)|^2 dz \wedge d\bar{z} < \infty$$

Let  $T$  be a bounded linear operator on  $L_a^2(\mathbb{D})$ , a subspace  $\mathfrak{M}$  of  $L_a^2(\mathbb{D})$  is called a *reducing subspace* of  $T$  if  $T(\mathfrak{M}) \subseteq \mathfrak{M}$  and  $T^*(\mathfrak{M}) \subseteq \mathfrak{M}$ . A reducing subspace  $\mathfrak{M}$  of  $T$  is called *minimal* if for every reducing subspace  $\tilde{\mathfrak{M}}$  of  $T$  such that  $\tilde{\mathfrak{M}} \subseteq \mathfrak{M}$  then either  $\tilde{\mathfrak{M}} = \mathfrak{M}$  or  $\tilde{\mathfrak{M}} = 0$ .

An  $n$ -th order Blaschke product  $B$  is the analytic function on  $\mathbb{D}$  given by

$$B(z) = e^{i\theta} \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z},$$

where  $\theta$  is a real number and  $a_i \in \mathbb{D}$  for  $1 \leq i \leq n$ .

Let  $T_B$  be the multiplication operator on  $L_a^2(\mathbb{D})$  by  $B$ . Zhu first studied the reducing subspaces of  $T_B$ , and showed that  $T_B$  has exactly two distinct minimal

reducing subspaces when the Blaschke product is of order 2 (cf. [16]). Motivated by this fact, Zhu conjectured that the number of minimal reducing subspaces of  $T_B$  equals to the order of  $B$  (cf. [16]). Guo, Sun, Zheng and Zhong showed that in general this is not true (cf. [12]), and they found that the number of minimal reducing subspaces of  $T_B$  equals to the number of connected components of the Riemann surface of  $B(z) = B(w)$  when the order of  $B$  is 3,4,6. Then they conjectured that the number of minimal reducing subspaces of  $T_B$  equals to the number of connected components of the Riemann surface of  $B(z) = B(w)$  for any finite Blaschke product (the refined Zhu's conjecture, cf. [12]).

Set  $B(z) := e^{i\theta} \prod_{i=1}^n \frac{z-a_i}{1-\bar{a}_i z} = \frac{P(z)}{Q(z)}$  where  $P(z) = e^{i\theta} \prod_{i=1}^n (z - a_i)$  and  $Q(z) = \prod_{i=1}^n (1 - \bar{a}_i z)$ , and let  $(\mathcal{W}^*(T_B))'$  denote the *double commutant* of  $T_B$ , i.e., all  $T$  in  $\mathcal{L}(L_a^2(\mathbb{D}))$  such that  $TT_B = T_B T$  and  $TT_B^* = T_B^* T$ . It is easy to see  $(\mathcal{W}^*(T_B))'$  is a Von Neumann algebra. The first author, Sun and Zheng proved following theorem in [10].

**Theorem 1.** *If  $B$  is a finite Blaschke product, then  $(\mathcal{W}^*(T_B))'$  has dimension  $q$ , where  $q$  is the number of connected components of the Riemann surface  $S_f$ , where  $f(z, w) = P(w)Q(z) - P(z)Q(w)$ .*

Zhu's refined conjecture is then proved once one can prove  $(\mathcal{W}^*(T_B))'$  is commutative, in the same paper, they proved affirmatively for the cases  $n \leq 8$ . Putinar, Wang and the first author [9] proved the following theorem for the general cases by using the notion of local inverses of a finite Blaschke product which is introduced by Thompson [15].

**Theorem 2.**  *$(\mathcal{W}^*(T_B))'$  is commutative for any finite Blaschke product.*

One can show that the restrictions of  $T_B$  onto its minimal reducing subspaces are not unitary equivalent to each other. Then the following question comes naturally: what are these operators obtained by restricting  $T_B$  onto its minimal reducing subspaces?

Firstly let us consider the special case when  $B(z) = z^n$ . For  $0 \leq i < n$ , set

$$\mathcal{L}_{n,i} = \vee \{z^l : l = i \bmod n\}.$$

$\{\mathcal{L}_{n,i}\}_{i=0}^{n-1}$  are minimal reducing subspaces of  $T_{z^n}$  and they are pairwise orthogonal. Set  $L_a^{2,i}(\mathbb{D} \setminus \{0\}) = L_a^2(\mathbb{D} \setminus \{0\}, |\frac{1}{z^{\frac{n-i-1}{n}}}|^2 dz \wedge d\bar{z})$ ,  $0 \leq i < n$ . Define the map  $U_i : L_a^{2,i}(\mathbb{D} \setminus \{0\}) \rightarrow \mathcal{L}_{n,i}$  as follows;

$$U_i(f)(z) = \sqrt{n} z^i f(z^n), \text{ for } z \in \mathbb{D}.$$

Then  $U_i$  is a unitary map such that  $U_i \circ T_z = T_{z^n}|_{\mathcal{L}_{n,i}} \circ U_i$ . Thus restrictions of  $T_{z^n}$  to its minimal reducing subspaces are unitarily equivalent to the multiplication operators  $T_z$  on weighted Bergman spaces, and there exists a distinguished minimal reducing subspace such that the restriction operator is unitarily equivalent to the Bergman shift. Further, one can calculate the curvatures of these restrictions, and prove that they are not unitarily equivalent, (c.f.[8])

We are interested in describing model spaces for the minimal reducing subspaces of Blaschke product  $B$  of order  $n$ . We show they are weighted Bergman spaces

attained from the analogue of bundle shifts building by using the Bergman space, and such that the multiplication operators on them are unitarily equivalent to restrictions of  $T_B$  on its minimal reducing subspaces. The goal of this paper is trying to provide generalized Bergman bundle shifts as the model spaces viz, complex geometry and Riemann surface, etc.

To describe the progress in this direction, we need to introduce some notations. For the *flat unitary vector bundle*  $E$  we denote the multiplication operator on the space of holomorphic  $L^2$  sections of  $E$ ,  $L_a^2(E)$ , by  $T_E$ . Set,  $\mathcal{S} = B(\{\beta \in \mathbb{D} : B'(\beta) = 0\})$ . One of our main result is the following.

**Theorem 3.** *Let  $B$  be a finite Blaschke product of order  $n$ . Let  $T_B$  be the multiplication operator on  $L_a^2(\mathbb{D})$  by  $B$ . Let  $E_B$  be the flat unitary vector bundle over  $\mathbb{D} \setminus \mathcal{S}$  determined by  $B$ . Let  $T_{E_B}$  be the multiplication operator on  $L_a^2(E_B)$  defined by  $z$ . Then the operator  $T_B$  is unitarily equivalent to the operator  $T_{E_B}$ .*

The definition of  $E_B$  will be explained in the next section.

## 2. GENERALIZED BUNDLE SHIFTS

For supplying the model spaces of  $T_B$  on its reducing subspaces, the first thing of this section is to define a generalized bundle shift and then classify them under unitary equivalence. The investigation of this section extends the paper [1].

**2.1. Generalized Bergman Bundle Shifts.** Let  $\Omega$  be an open subset of  $\mathbb{C}^m$ . Let  $\mathfrak{K}$  be a Hilbert space.

**Definition 4.** A *continuous vector bundle*, which is a family of Hilbert spaces over  $\Omega$ , is a topological space  $E$  together with:

- (1) A continuous map  $q : E \rightarrow \Omega$ ,
- (2) A Hilbert space structure on each fibre  $E_z = q^{-1}(z)$ ,  $z \in \Omega$ , such that the Hilbert space topology on  $E_z$  is the same as the topology inherited from  $E$ ,
- (3) For each  $z \in \Omega$ , there exists a neighborhood  $U$  of  $z$  in  $\Omega$  and a homeomorphism  $\phi_U : q^{-1}(U) \rightarrow U \times \mathfrak{K}$  such that:
  - (a) For each  $(w, k) \in U \times \mathfrak{K}$ , the point  $\phi_U(w, k) \in E_w$ ,
  - (b) For each  $w \in U$ , the map  $\phi_U^w : \mathfrak{K} \rightarrow E_w$  defined by  $\phi_U^w(k) = \phi_U(w, k)$  is a continuous linear isomorphism.

**Definition 5.** (1) A continuous vector bundle  $E$  is called a *holomorphic vector bundle* if for all open sets  $U$  and  $V$  such that  $U \cap V \neq \emptyset$  then

$$\phi_U \circ \phi_V^{-1}|_{(U \cap V) \times \mathfrak{K}} : (U \cap V) \times \mathfrak{K} \rightarrow (U \cap V) \times \mathfrak{K}$$

is given by the following expression

$$\phi_U \circ \phi_V^{-1}(w, k) = (w, \phi_{UV}(w)k),$$

where  $\phi_{UV} : U \cap V \rightarrow \mathcal{GL}(\mathfrak{K})$  is holomorphic and  $\mathcal{GL}(\mathfrak{K})$  is the set of invertible bounded linear operator on the Hilbert space  $\mathfrak{K}$ .

- (2) A holomorphic vector bundle  $E$  is called a *Hermitian holomorphic vector bundle* if an inner product is given on  $E_z$  which varies smoothly as a function of  $z$ , that is, for given any two smooth local sections  $s, t$  of  $E$ , the function  $z \mapsto \langle s(z), t(z) \rangle_z$  is a smooth function.

For a given Hermitian holomorphic vector bundle  $E$ , let  $\Gamma_a(\Omega)$  be the space of holomorphic sections of  $E$  over  $\Omega$ . The Bergman space  $L_a^2(E)$  is defined as

$$L_a^2(E) = \{f \in \Gamma_a(\Omega) : \|f\|_{L_a^2(E)}^2 := \int_{\Omega} \|f(z)\|_{E_z}^2 dV(z) < \infty\}.$$

Usually, a reproducing kernel Hilbert space consists of vector-valued functions taking value in  $\mathbb{C}^k$ , but more general notion is possible. A point evaluation  $ev_z : L_a^2(E) \rightarrow E_z$  is a continuous function for all  $z \in \Omega$ , so  $L_a^2(E)$  is a reproducing kernel Hilbert space. As a fact, the function  $K_E : \Omega \times \Omega \rightarrow \mathcal{L}(E^*, E)$  defined by  $K_E(z, w) = ev_z \circ ev_w^*$ , for all  $z, w \in \Omega$ , takes value in  $\mathcal{L}(E_w^*, E_z)$ , and for  $\eta \in E_z^*$  and  $\xi \in E_w^*$  we have

$$\begin{aligned} \langle \eta, K_E(z, w)\xi \rangle_{E_z} &= \langle \eta, ev_z \circ ev_w^* \xi \rangle_{E_z} = \langle ev_z^* \eta, ev_w^* \xi \rangle_{L_a^2(E)} \\ &= \langle ev_w \circ ev_z^* \eta, \xi \rangle_{E_w} = \langle K_E(w, z)\eta, \xi \rangle_{E_w} \\ &= \langle \eta, K_E(w, z)^* \xi \rangle_{E_z}, \end{aligned}$$

which implies

$$K_E(z, w) = K_E(w, z)^*.$$

Moreover,

$$\langle f, K_E(\cdot, w)\xi \rangle_{L_a^2(E)} = \langle f, ev_w^* \xi \rangle_{L_a^2(E)} = \langle ev_w(f), \xi \rangle_{E_w} = \langle f(w), \xi \rangle_{E_w}.$$

Hence,  $K_E$  is the reproducing kernel of  $L_a^2(E)$ .

There is another way to view the Bergman space over the flat vector bundle. In particular, one can view it as a space of functions on the universal covering space. Sometime this representation is quite useful.

**Definition 6.** Let  $E$  be a vector bundle. A *unitary coordinate covering* for  $E$  is a coordinate covering  $\{U, \phi_U\}$  such that for each open set  $U$  and  $z \in U$ , the fiber map  $\phi_U|_{E_z} : E_z \rightarrow \{z\} \times \mathfrak{K}$  is unitary. The unitary coordinate covering  $\{U, \phi_U\}$  is said to be *flat* if the matrix valued functions  $\phi_{UV} : U \cap V \rightarrow \mathcal{U}(\mathfrak{K})$  are constants, where  $\phi_U \circ \phi_V^{-1}(z, v) = (z, \phi_{UV}(z)v)$  for  $z \in U \cap V$  and  $v \in \mathfrak{K}$ . A *flat* unitary vector bundle is a vector bundle equipped with a flat unitary coordinate covering.

A flat structure on a vector bundle over  $\Omega$  gives rise to a representation

$$\alpha : \pi_1(\Omega) \rightarrow \mathcal{U}(\mathfrak{K})$$

of the fundamental group of  $\Omega$  via parallel displacement. Conversely, suppose we have a representation  $\alpha$  and let  $\tilde{\Omega}$  be the universal covering space of  $\Omega$ , we can construct a flat bundle as follows: the equivalence relation  $(z_1, v_1) \sim (z_2, v_2)$  on  $\tilde{\Omega} \times \mathfrak{K}$  is defined as if  $z_2 = A(z_1)$  and  $v_2 = \alpha(A)v_1$  for some  $A \in \pi_1(\Omega)$ . This equivalence relation gives rise to a flat vector bundle  $E_\alpha = \tilde{\Omega} \times \mathfrak{K} / \sim$  with natural projection  $\pi$ . The vector bundle  $E_\alpha$  constructed above is a flat unitary vector bundle. Discussions about flat vector bundles can be found in [1, 3, 14]. The group  $\mathcal{U}(\mathfrak{K})$  acts on  $\text{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$  via conjugation  $(V, \alpha) \rightarrow V\alpha V^*$ . The set of equivalence classes is denoted by  $\text{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))/\mathcal{U}(\mathfrak{K})$ .

**Proposition 7.** ([1]) *There is one to one correspondence between  $\text{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))/\mathcal{U}(\mathfrak{K})$  and the set of equivalence classes of flat unitary vector bundles over  $\Omega$ .*

For a flat unitary vector bundle  $E$ , the  $m$ -tuple multiplication operator  $(T_{z_1}, \dots, T_{z_m})$  on  $L_a^2(E)$  is denoted by  $T_E$ .

**Definition 8.** Let  $X$  and  $Y$  be two topological spaces. A continuous map  $q : Y \rightarrow X$  is a *covering map* if there is a neighborhood  $U$  for each point  $x \in X$  such that;

- (1) the set  $q^{-1}(U)$  is a disjoint union of open sets  $V_i$ ,
- (2) the restriction map of  $q$  to  $V_i$  is a homeomorphism of  $V_i$  onto  $U$ .

The group of covering transformations for  $q$  is the group of homeomorphisms  $A$  of  $Y$  such that  $q \circ A = q$ . Two covering spaces  $(Y_1, q_1)$  and  $(Y_2, q_2)$  of  $X$  are said to be *equivalent* if there is a homeomorphism  $\phi$  from  $Y_1$  onto  $Y_2$  such that  $q_2 \circ \phi = q_1$ .

Let  $q : \tilde{\Omega} \rightarrow \Omega$  be a universal covering map. Let  $J_q(z) = ((\frac{\partial q_i}{\partial z_j}(z)))_{i,j=1}^m$  be Jacobin of  $q = (q_1, \dots, q_m)$ . For  $\alpha \in \text{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$ ,  $L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$  is defined to be space of functions  $f$  in  $L_a^2(\tilde{\Omega}, \mathfrak{K}, |\det J_q(z)|^2 dV(z))$  such that

$$\int_{\tilde{\Omega}} \|f(z)\|_{\mathfrak{K}}^2 |\det J_q|^2 dV(z) < \infty$$

and  $f(A(z)) = \alpha(A)f(z)$  for all  $A \in \pi_1(\Omega)$  and  $z \in \tilde{\Omega}$ . A bounded holomorphic function  $\phi$  defined on  $\tilde{\Omega}$  is said to be  $\pi_1(\Omega)$ -*automorphic* if  $\phi \circ A = \phi$  for each  $A$  in  $\pi_1(\Omega)$ . Let  $H^\infty(\tilde{\Omega}, \pi_1(\Omega))$  denote the set of all  $\pi_1(\Omega)$ -automorphic bounded holomorphic functions. The space  $L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$  is invariant for  $H^\infty(\tilde{\Omega}, \pi_1(\Omega))$ , that is, if  $\phi \in H^\infty(\tilde{\Omega}, \pi_1(\Omega))$  and  $f \in L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$  then  $\phi f \in L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$ . In particular, it is invariant for multiplication by the components  $q_i$ ,  $1 \leq i \leq m$ , of covering map  $q$  and we define the operator  $T_\alpha = (T_{\alpha_1}, \dots, T_{\alpha_m})$  on  $L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$  by  $T_{\alpha_i}(f) = q_i f$ , for  $1 \leq i \leq m$ .

**Proposition 9.** If  $\alpha$  is in  $\text{Hom}(\pi_1(\Omega), \mathcal{U}(\mathfrak{K}))$  and  $E_\alpha$  is the flat unitary bundle over  $\Omega$  determined by  $\alpha$ , then  $T_\alpha$  is unitarily equivalent to the bundle shift  $T_{E_\alpha}$ .

*Proof.* Define a map  $F : L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z)) \rightarrow L_a^2(E_\alpha)$  as follows

$$F(f)(w) = [(z, f(z))] \quad \text{for } z \in q^{-1}(w), w \in \Omega,$$

where  $[(z, f(z))]$  denote equivalence class of  $(z, f(z))$ . It is easy to see that

$$\|F(f)(w)\|_{(E_\alpha)_w}^2 = \|f(z)\|_{\mathfrak{K}}^2.$$

(1)  $F$  is an isometry:

$$\begin{aligned} \|F(f)\|_{L_a^2(E)}^2 &= \int_{\Omega} \|F(f)(w)\|_{(E_\alpha)_w}^2 dV(w) \\ &= \int_{\tilde{\Omega}} \|F(f)(q(z))\|_{(E_\alpha)_w}^2 |\det J_q(z)|^2 dV(z) \\ &= \int_{\tilde{\Omega}} \|f(z)\|_{\mathfrak{K}}^2 |\det J_q(z)|^2 dV(z) \\ &= \|f\|_{L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))}^2 \end{aligned}$$

(2)  $F$  is onto: Let  $s$  be a section of  $E_\alpha$  such that  $s \in L_a^2(E_\alpha)$ . Define function  $f_s : \mathbb{D} \rightarrow \mathfrak{K}$  such that  $s(q(z)) = [(z, f_s(z))]$  for  $z \in \tilde{\Omega}$ . One can show that  $f_s$  is

holomorphic and  $q(z) = q(A(z))$  implies that  $f_s(A(z)) = \alpha(A)f_s(z)$  for all  $A \in \pi_1(\Omega)$  and  $z \in \tilde{\Omega}$ . One easily see that

$$\|s\|_{L_a^2(E_\alpha)}^2 = \int_{\tilde{\Omega}} \|f_s(z)\|_{\mathfrak{K}}^2 |\det J_q(z)|^2 dV(z).$$

Hence  $f_s \in L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$  and  $F(f_s) = s$ .

(3)  $F$  intertwines  $T_\alpha$  and  $T_{E_\alpha}$ : for  $f \in L_a^2(\tilde{\Omega}, \mathfrak{K}, \alpha, |\det J_q|^2 dV(z))$ ,  $w \in \Omega$ ,  $z \in q^{-1}(w)$  and  $1 \leq i \leq m$

$$\begin{aligned} (F \circ T_{\alpha_i})(f)(w) &= F(q_i f)(w) = [(z, (q_i f)z)] = [(z, q_i(z)f(z))] \\ &= [(z, w_i f(z))] = w_i[z, f(z)] = w_i F(f)(w) \\ &= (T_{z_i} \circ F)(f)(w). \end{aligned}$$

Thus  $F \circ T_{\alpha_i} = T_{z_i} \circ F$ . □

**2.2. Bergman Bundle Shifts.** Although most of results in this part can be extended to multivariable cases or weighted Bergman spaces, we restrict ourself to one variable case for further use. Moreover, when the domain has nice boundary, we can follow the approach of Abrahamse and the the first author which we do in this section. In section 2.4, we show how to provide an alterative approach covering the general case.

Let  $R$  be a multiply connected domain in  $\mathbb{C}$  whose boundary consists of  $n+1$  analytic Jordan curves and  $\tilde{q} : \mathbb{D} \rightarrow R$  be the covering map. Let  $G$  be the group of covering transformations of  $\tilde{q}$ . The group  $G$  is isomorphic to fundamental group of  $R$ .

A function  $f$  on  $\mathbb{D}$  is said to be  $G$ -automorphic if  $f \circ A = f$  for each  $A$  in  $G$ . Let  $L^2(\mathbb{D}, \mathbb{C}^n, G)$  be a subspace of  $G$ -automorphic functions in  $L^2(\mathbb{D}, \mathbb{C}^n)$  and let  $L_a^2(\mathbb{D}, \mathbb{C}^n, G)$  be a subspace of  $G$ -automorphic functions in  $L_a^2(\mathbb{D}, \mathbb{C}^n)$ . Let  $L^\infty(\mathbb{D}, G)$  be the algebra of  $G$ -automorphic functions in  $L^\infty(\mathbb{D})$  and let  $H^\infty(\mathbb{D}, G)$  be the algebra of  $G$ -automorphic functions in  $H^\infty(\mathbb{D})$ .

**Proposition 10.** (1) *The smallest invariant subspace for  $L^\infty(\mathbb{D}, G)$  containing  $L_a^2(\mathbb{D}, \mathbb{C}^n, G)$  is  $L^2(\mathbb{D}, \mathbb{C}^n, G)$ , where  $L^\infty(\mathbb{D}, G)$  acts on  $L^2(\mathbb{D}, \mathbb{C}^n, G)$  pointwisely.*

(2) *The smallest invariant subspace for  $H^\infty(\mathbb{D})$  containing  $L_a^2(\mathbb{D}, \mathbb{C}^n, G)$  is  $L_a^2(\mathbb{D}, \mathbb{C}^n)$ , where  $H^\infty(\mathbb{D})$  acts on  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  pointwisely.*

(3) *The smallest invariant subspace for  $L^\infty(\mathbb{D})$  containing  $L_a^2(\mathbb{D}, \mathbb{C}^n, G)$  is  $L^2(\mathbb{D}, \mathbb{C}^n)$ , where  $L^\infty(\mathbb{D})$  acts on  $L^2(\mathbb{D}, \mathbb{C}^n)$  pointwisely.*

*Proof.* Since  $L_a^2(\mathbb{D}, \mathbb{C}^n, G)$  contains the constant functions, the proof is immediate. □

An operator  $T$  on  $L_{\mathbb{C}^n}^2(\mathbb{D})$  will be called to be *decomposable* if there is a weakly measurable function  $w \mapsto T_w$  from  $\mathbb{D}$  to  $B(\mathbb{C}^n)$  such that for  $f \in L_{\mathbb{C}^n}^2(\mathbb{D})$ ,  $(Tf)(w) = T_w f(w)$ , a.e.  $dv$ . It is well known that an operator on  $L_{\mathbb{C}^n}^2(\mathbb{D})$  is decomposable if and only if it commutes with  $L^\infty(\mathbb{D})$ . Moreover, the algebraic operations can be performed pointwisely, that is  $[(T+S)f](w) = T_w f(w) + S_w f(w)$ ,  $(T^*f)(w) = T_w^* f(w)$ , etc., and the norm is  $\|T\| = \text{ess sup } \|T_w\|$ . One can see [6, 7] for further references.



**Proposition 11.** *For every unitary representation  $\alpha$  of  $G$  on  $\mathbb{C}^n$ , there is a decomposable operator  $\Phi_\alpha$  on  $L^2_{\mathbb{C}^n}(\mathbb{D})$  such that  $\Phi_\alpha(L^2_a(\mathbb{D}, \mathbb{C}^n, G)) = L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ . Moreover,  $\Phi_\alpha$  satisfies  $\Phi_{\alpha, g(\lambda)} = \alpha(g)\Phi_{\alpha, \lambda}$  a.e.  $dm$  for every  $g \in G$ .*

*Proof.* Note that the covering map  $q : \mathbb{D} \rightarrow R$  can be extended to a covering map from an open set  $\tilde{\mathbb{D}}$  containing  $\mathbb{D}$  to an open set  $\tilde{R}$  containing the closure of  $R$ . Given  $\alpha$ , let  $E_\alpha$  be the flat unitary bundle over  $\tilde{\mathbb{D}}$ , and let  $\pi$  denote the projection from  $\tilde{\mathbb{D}} \times \mathbb{C}^n$  onto  $E_\alpha$ .  $E_\alpha$  is analytically equivalent to the trivial bundle  $\tilde{R} \times \mathbb{C}^n$ , so there exists an analytic isomorphism  $\phi$  from  $E_\alpha$  to  $\tilde{R} \times \mathbb{C}^n$ . It is clear that the composition  $\phi \circ \pi$  is a bundle map from  $\tilde{\mathbb{D}} \times \mathbb{C}^n$  onto  $\tilde{R} \times \mathbb{C}^n$  which yields an analytic function  $\Phi$  from  $\tilde{\mathbb{D}}$  to  $\mathcal{GL}(\mathbb{C}^n)$ , the set of all invertible operators on  $\mathbb{C}^n$ , such that  $\Phi(g(z)) = \alpha(g)\Phi(z)$  for all  $z \in \tilde{\mathbb{D}}$  and  $g \in G$ . Let  $\Phi_\alpha$  be the restriction of  $\Phi$  to the closed disk, it follows that there is a  $K > 0$  such that  $|\Phi_\alpha(z)| \leq K$  and  $|\Phi_\alpha(z)^{-1}| \leq K$  for all  $z$  in the closed disk. Let  $f \in L^2_{\mathbb{C}^n}(\mathbb{D})$ , consider

$$\begin{aligned} \int_{\mathbb{D}} |(\Phi_\alpha f)(z)|^2 dm &= -\frac{1}{2\pi i} \int_{\mathbb{D}} |(\Phi_\alpha f)(z)|^2 dz \wedge d\bar{z} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{D}} |\Phi_\alpha(z)f(z)|^2 dz \wedge d\bar{z} \\ &\leq -\frac{K}{2\pi i} \int_{\mathbb{D}} |f(z)|^2 dz \wedge d\bar{z} \\ &= K \|f\|_{L^2_{\mathbb{C}^n}(\mathbb{D})}^2 \end{aligned}$$

Hence,  $\Phi_\alpha$  is an invertible decomposable operator on  $L^2_{\mathbb{C}^n}(\mathbb{D})$ . Since  $\Phi_\alpha$  is analytic and  $\Phi_\alpha(z)$  and  $\Phi_\alpha(z)^{-1}$  are uniformly bounded for  $z \in \tilde{\mathbb{D}}$ , so we have  $\Phi_\alpha(L^2_a(\mathbb{D}, \mathbb{C}^n, G)) = L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$ .  $\square$

**Proposition 12.** (1) *The smallest invariant subspace for  $L^\infty(\mathbb{D}, G)$  containing  $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$  is  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ .*  
 (2) *The smallest invariant subspace for  $H^\infty(\mathbb{D})$  containing  $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$  is  $L^2_a(\mathbb{D}, \mathbb{C}^n)$ .*  
 (3) *The smallest invariant subspace for  $L^\infty(\mathbb{D})$  containing  $L^2_a(\mathbb{D}, \mathbb{C}^n, \alpha)$  is  $L^2(\mathbb{D}, \mathbb{C}^n)$ .*

*Proof.* The proof follows from Proposition 10 and Proposition 11.  $\square$

**Lemma 13.** *An operator on  $L^2(\mathbb{D}, \mathbb{C}^n, G)$  commutes with  $L^\infty(\mathbb{D}, G)$  if and only if it is a  $G$ -automorphic decomposable operator.*

*Proof.* The comnutant of  $L^\infty(\Omega)$  defined on  $L^2(\Omega, \mathbb{C}^n)$  consists of decomposable operators on  $L^2(\Omega, \mathbb{C}^n)$  [6, 7]. Since  $L^2(\mathbb{D}, \mathbb{C}^n, G)$  is isometrically isomorphic to  $L^2(\Omega, \mathbb{C}^n)$ , the lemma follows.  $\square$

**Proposition 14.** *An operator  $W : L^2(\mathbb{D}, \mathbb{C}^n, \alpha) \rightarrow L^2(\mathbb{D}, \mathbb{C}^n, \beta)$  intertwines  $L^\infty(\mathbb{D}, G)$  if and only if  $W$  is decomposable and  $W_{g(\lambda)} = \beta(g)W_\lambda \alpha(g)^*$  a.e.  $dv$  for all  $g \in G$ .*

*Proof.* Given  $W$ , the operator  $(\Phi_\beta)^{-1}W\Phi_\alpha$  on  $L^2(\mathbb{D}, \mathbb{C}^n, G)$  commutes with  $L^\infty(\mathbb{D}, G)$ . Thus, by the lemma, there is a  $G$ -automorphic decomposable operator  $V$  such that

$V = (\Phi_\beta)^{-1}W\Phi_\alpha$ . Hence  $W = \Phi_\beta V(\Phi_\alpha)^{-1}$  and thus  $W$  is decomposable. Moreover,

$$\begin{aligned} W_{g(\lambda)} &= (\Phi_{\beta,g(\lambda)})V_{g(\lambda)}(\Phi_{\alpha,g(\lambda)})^{-1} \\ &= \beta(g)\Phi_{\beta,\lambda}V_\lambda(\alpha(g)\Phi_{\alpha,\lambda})^{-1} \\ &= \beta(g)(\Phi_{\beta,\lambda}V_\lambda(\Phi_{\alpha,\lambda})^{-1})(\alpha(g))^{-1} \\ &= \beta(g)W_\lambda\alpha(g)^*. \end{aligned}$$

The sufficient condition is obvious.  $\square$

We state a series of lemmas, culminating in characterization of reducing subspaces of Bergman bundle shifts.

**Lemma 15.** *If a decomposable operator  $T$  on  $L^2(\mathbb{D}, \mathbb{C}^n)$  such that  $T(L_a^2(\mathbb{D}, \mathbb{C}^n)) \subseteq L_a^2(\mathbb{D}, \mathbb{C}^n)$  then  $F : \mathbb{D} \rightarrow \mathbb{C}^n$ , defined as  $F(\lambda) = T_\lambda u$ , is a holomorphic map, for each fixed  $u \in \mathbb{C}^n$ .*

*Proof.* For  $u \in \mathbb{C}^n$ , consider the constant function  $f(\lambda) = u$  for all  $\lambda \in \mathbb{D}$ . Clearly,  $f \in L_a^2(\mathbb{D}, \mathbb{C}^n)$ . By the given condition,  $T(f) \in L_a^2(\mathbb{D}, \mathbb{C}^n)$ . In particular,  $T(f)$  is a holomorphic function on  $\mathbb{D}$ . For any  $\lambda_0 \in \mathbb{D}$ ,  $\lim_{\lambda \rightarrow \lambda_0} \frac{T(f)(\lambda) - T(f)(\lambda_0)}{\lambda - \lambda_0}$  exists. Now,

$$\begin{aligned} \frac{T(f)(\lambda) - T(f)(\lambda_0)}{\lambda - \lambda_0} &= \frac{T_\lambda(f(\lambda)) - T_{\lambda_0}(f(\lambda_0))}{\lambda - \lambda_0} \\ &= \frac{T_\lambda(u) - T_{\lambda_0}(u)}{\lambda - \lambda_0} \\ &= \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}. \end{aligned}$$

Hence  $\lim_{\lambda \rightarrow \lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0}$  exists. Thus  $F$  is a holomorphic function on  $\mathbb{D}$ .  $\square$

**Lemma 16.** *If a decomposable operator  $T$  on  $L^2(\mathbb{D}, \mathbb{C}^n)$  such that  $T^*(L_a^2(\mathbb{D}, \mathbb{C}^n)) \subseteq L_a^2(\mathbb{D}, \mathbb{C}^n)$  then  $\tilde{F} : \mathbb{D} \rightarrow \mathbb{C}^n$ , defined as  $\tilde{F}(\lambda) = T_\lambda^* u$ , is a holomorphic map, for each fixed  $u \in \mathbb{C}^n$ .*

*Proof.* The proof is same as proof of Lemma 15.  $\square$

**Lemma 17.** *A decomposable operator  $T$  on  $L^2(\mathbb{D}, \mathbb{C}^n)$  commutes with the projection onto  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  if and only if it is a constant.*

*Proof.* It is sufficient clearly. On the other side, assume that  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  reduces  $T$ , that is,  $T(L_a^2(\mathbb{D}, \mathbb{C}^n)) \subseteq L_a^2(\mathbb{D}, \mathbb{C}^n)$  and  $T^*(L_a^2(\mathbb{D}, \mathbb{C}^n)) \subseteq (L_a^2(\mathbb{D}, \mathbb{C}^n))$ .

Fix  $u$  and  $v$  in  $\mathbb{C}^n$  and define  $h(\lambda) = \langle T_\lambda u, v \rangle$ . Since  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  is invariant for  $T$ , so by Lemma 15, the function  $h$  is holomorphic on  $\mathbb{D}$ . Similarly, since  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  is invariant for  $T^*$ , so by Lemma 16, the function  $\tilde{h}(\lambda) = \overline{h(\bar{\lambda})}$  is holomorphic on  $\mathbb{D}$ . Hence,  $h$  is a constant function on  $\mathbb{D}$ . Since  $u$  and  $v$  are arbitrary, so  $\lambda \mapsto T_\lambda$  is a constant function from  $\mathbb{D}$  to  $\mathcal{L}(\mathbb{C}^n)$  also.  $\square$

**Theorem 18.** *The operators  $T_\alpha$  and  $T_\beta$  are unitarily equivalent if and only if  $\alpha$  and  $\beta$  are unitarily equivalent.*

*Proof.* It is easy to check the sufficiency. Conversely, suppose that  $U$  is an isometry from  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  onto  $L_a^2(\mathbb{D}, \mathbb{C}^n, \beta)$  with  $UT_\alpha = T_\beta U$ . Then, by the fact that the minimal normal extension for a subnormal operator is unique, one can extend  $U$  to



a unitary equivalence of their minimal normal extensions. The minimal extension of  $T_\alpha$  is multiplication by  $q$  on  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ , and the extension of  $U$  is a unitary operator  $\tilde{U}$  from  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  to  $L^2(\mathbb{D}, \mathbb{C}^n, \beta)$ , and it intertwines  $L^\infty(\mathbb{D}, G)$ . Thus  $\tilde{U}$  is decomposable and  $\tilde{U}(L_a^2(\mathbb{D}, \mathbb{C}^n) = L_a^2(\mathbb{D}, \mathbb{C}^n)$ . Hence  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  reduces  $\tilde{U}$ , it is a constant. Thus there is a unitary operator  $V$  on  $\mathbb{C}^n$  such that  $\tilde{U}_\lambda = V$  a.e.  $dv$ . But  $V = \tilde{U}_{g(\lambda)} = \beta(g)\tilde{U}_\lambda\alpha(g)^* = \beta(g)V\alpha(g)^*$  a.e.  $dv$ , and thus  $\alpha(g) = V^*\beta(g)V$ , which shows that  $\alpha$  and  $\beta$  are unitarily equivalent.  $\square$

**Corollary 19.** *If  $E$  and  $E'$  are flat unitary bundles over  $\Omega$ , then operators  $T_E$  and  $T_{E'}$  are unitarily equivalent if and only if  $E$  and  $E'$  are equivalent as flat unitary bundles.*

*Proof.* First Suppose that the flat unitary vector bundles  $E$  and  $E'$  are equivalent. Let  $\Theta : E \rightarrow E'$  be the bundle isomorphism. Then, the map  $f(z) \mapsto \Theta(z)f(z)$  for  $z \in \Omega$  defines a module isomorphism between  $A(\Omega)$  modules  $\Gamma_a(E)$  and  $\Gamma_a(E')$ . For  $z \in \Omega$ ,  $\|\Theta(z)f(z)\|_{E'_z}^2 = \|f(z)\|_{E_z}^2$ . If  $f \in L_a^2(E)$  then

$$\begin{aligned} \|\Theta f\|_{L_a^2(E')}^2 &= -\frac{1}{2\pi i} \int_{\Omega} \|\Theta(z)f(z)\|_{E'_z}^2 dz \wedge d\bar{z} \\ &= -\frac{1}{2\pi i} \int_{\Omega} \|f(z)\|_{E_z}^2 dz \wedge d\bar{z} \\ &= \|f\|_{L_a^2(E)}^2. \end{aligned}$$

Hence  $\Theta$  defines an isometry from  $L_a^2(E)$  onto  $L_a^2(E')$ . Since  $\Theta$  is a module isomorphism, this intertwines  $T_E$  and  $T_{E'}$ .

Conversely suppose that operators  $T_E$  and  $T_{E'}$  are unitarily equivalent, so by Proposition 9 and Theorem 18, bundle  $E$  and  $E'$  are unitarily equivalent as flat unitary vector bundles.  $\square$

If  $\Omega_1$  and  $\Omega_2$  are two multiply connected domains in the complex plane whose boundaries consist of Jordan curves, and biholomorphic to each other by the map  $\varphi$ .  $E$  is a flat unitary bundle over  $\Omega_2$ , then the pull back bundle  $\varphi_1^*E$  is a flat unitary bundle over  $\Omega_1$ .

**Corollary 20.** *The bundle shifts  $T_E$  and  $T_{\varphi^*E}$  are unitarily equivalent.*

**2.3. The commutant of  $W^*(T_\alpha)$ .** We recall some results about the relation between subnormal operator and its normal extension (cf. [5]). Let  $S$  be a subnormal operator on a Hilbert space  $\mathcal{H}$  and let  $N$  be its minimal normal extension on the Hilbert space  $\mathcal{K}$ . If an operator commute with operator  $S$  and  $S^*$ , that is, the operator in the commutant of the  $W^*$  algebra  $\mathcal{W}^*(S)$  generated by  $S$ , lifts to operators that commute with  $N$  (cf. [2]).

Let  $P : \mathcal{K} \rightarrow \mathcal{H}$  be the orthogonal projection. Let  $\mathcal{A}$  denote the algebra of operators on  $\mathcal{K}$  that commute with  $N$  and  $P$ , and let  $(\mathcal{W}^*(S))'$  denote commutants of  $\mathcal{W}^*(S)$ .

**Theorem 21.** *If  $E$  is a  $n$ -dimensional flat unitary bundle over  $\Omega$ , then the algebra of commutants of  $T_E$  is  $H_{L(E)}^\infty(\Omega)$ .*

*Proof.* Every analytic vector bundle over  $\Omega$  is analytically trivial by Grauert theorem, and we assume that  $V : \Omega \times \mathbb{C}^n \rightarrow E$  is an analytic isomorphism. Then  $V$  defines the similarity  $\tilde{V} : L_a^2(E) \rightarrow L_a^2(\Omega, \mathbb{C}^n)$  between  $T_{\mathbb{C}^n}$  and  $T_E$ , that is,  $T_E = \tilde{V}^{-1}T_{\mathbb{C}^n}\tilde{V}$ . If  $L_a^2(\Omega, \mathbb{C}^n)$  is rewritten as  $L_a^2(\Omega) \otimes \mathbb{C}^n$ , then  $T_{\mathbb{C}^n} = T_z \otimes I$ , where  $I$  is the identity operator on  $\mathbb{C}^n$ .  $A$  is an operator on  $L_a^2(E)$  commuting with  $T_E$ , then

$$A\tilde{V}^{-1}(T_z \otimes I)\tilde{V} = AT_E = T_EA = \tilde{V}^{-1}(T_z \otimes I)\tilde{V}A.$$

This means  $(T_z \otimes I)(\tilde{V}^{-1}A\tilde{V}) = (\tilde{V}^{-1}A\tilde{V})(T_z \otimes I)$ , so  $(\tilde{V}^{-1}A\tilde{V}) \in (T_z \otimes I)' = H^\infty(\Omega) \otimes M_n(\mathbb{C}^n)$ . It follows that  $A \in \tilde{V}(H^\infty(\Omega) \otimes M_n(\mathbb{C}^n))\tilde{V}^{-1} = H_{L(E)}^\infty(\Omega)$ .

Conversely, it is clear that every  $\Phi \in H_{L(E)}^\infty(\Omega)$  commutes with  $T_E$ , and the proof is completed.  $\square$

**Theorem 22.** [2] *The map  $A \mapsto A|_{\mathcal{H}}$  is a  $*$ -isometric isomorphism from  $\mathcal{A}$  onto  $(\mathcal{W}^*(S))'$ .*

Let  $\mathcal{W}^*(\alpha)$  be the  $W^*$ -subalgebra of  $\mathcal{L}(\mathbb{C}^n)$  generated by  $\alpha(G)$ .

**Theorem 23.** *There is a  $*$ -isometric isomorphism from  $(\mathcal{W}^*(\alpha))'$  onto  $(\mathcal{W}^*(T_\alpha))'$ .*

*Proof.* Let  $\mathcal{U}$  be the algebra of operators on  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  that commute with  $L^\infty(\mathbb{D}, G)$  and with the projection onto  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . For  $B$  in  $(\mathcal{W}^*(\alpha))'$  and  $f$  in  $L^2(\mathbb{C}^n, \alpha)$ , observe that  $(Bf) \circ A = B(f \circ A) = B(\alpha(A)f) = \alpha(A)Bf$ , for all  $A$  in  $G$ . Thus, there is a decomposable  $\tau_\alpha(B)$  on  $L^2(\mathbb{D}, \mathbb{C}^n)$  defined by  $\tau_\alpha(B)(f) = Bf$ .

Since a constant decomposable operator maps  $L_a^2(\mathbb{D})$  into  $L_a^2(\mathbb{D})$  and since  $\tau_\alpha(B)$  maps  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  into  $L^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ , thus it also maps  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  into  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . In other words, it takes  $(\mathcal{W}^*(\alpha))'$  into  $\mathcal{U}$ . Clearly,  $\tau_\alpha(B^*)f = B^*f = (\tau_\alpha B)^*f$  and  $\|\tau_\alpha B\| = \text{ess sup}_\lambda \|(\tau_\alpha(B))_\lambda\| = \|B\|$ . Thus  $\tau_\alpha$  is  $*$ -isometric map. If  $X$  is in  $\mathcal{U}$ , then by Proposition 14, the operator  $X$  is decomposable  $X_{g(\lambda)} = \alpha(g)X_\lambda\alpha(g)^*$  for all  $g \in G$ . Since  $X$  is reduced by  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ , so  $X(L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)) \subseteq L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$  and  $X^*(L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)) \subseteq L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . Thus, both  $X$  and  $X^*$  maps  $L_a^2(\mathbb{D}, \mathbb{C}^n)$  to  $L_a^2(\mathbb{D}, \mathbb{C}^n)$ . Hence by Lemma 17,  $X$  is a decomposable operator, and so it is a constant, thus there is a  $B$  in  $\mathcal{L}(\mathbb{C}^n)$  with  $X_\lambda = B$  a.e.  $dm$ . Now,

$$B = X_{g(\lambda)} = \alpha(g)X_\lambda\alpha(g)^* = \alpha(g)B\alpha(g)^*.$$

Thus operator  $B$  is in  $(\mathcal{W}^*(\alpha))'$  and so  $X = \tau_\alpha(B)$ . This proves that  $\tau_\alpha$  is onto.  $\square$

Note that the algebra  $\mathcal{W}^*(T_\alpha)'$  is finite dimensional for a Bergman bundle shift even though the Bergman space  $L_a^2(E)$  is infinite dimensional.

**2.4. The case of bad boundary.** In section 3, we need similar results about bundle shifts over multiply connected domains to  $D_0 = \mathbb{D} \setminus \mathcal{S}$  obtained by removing a finite subset from  $\mathbb{D}$ . Although, the proceeding results don't apply directly. In this case, the Bergman bundle shift is a Cowen-Douglas operator on  $D_0$ , we can prove similar results by applying ideas from [4]. In particular,  $D_0$  is a multiply connected domain with fundamental group  $\pi_1(D_0)$ , and its universal covering is  $\mathbb{D}$ .  $\mathbb{D} \times \mathbb{C}^n$  is the trivial bundle over  $\mathbb{D}$ ,  $E_\alpha$  is the corresponding flat unitary vector bundle over  $D_0$  defined by a representation  $\alpha : \pi_1(D_0) \rightarrow \mathcal{U}_n$ . The Bergman space  $L_a^2(E, \alpha)$  of holomorphic sections on  $E_\alpha$  is defined similarly. It can be characterized

as the space of all  $\alpha$ -automorphic functions in  $L_a^2(\mathbb{D}, \mathbb{C}^n)$ , denoted by  $L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . And we define the Bergman bundle shift  $T_\alpha$  also. For  $D_0$ , we have similar arguments and establish results which are similar to Theorem 18.

**Theorem 24.** *There is a onto  $*$ -isometric isomorphism between  $(\mathcal{W}^*(T_\alpha))'$  and  $(\mathcal{W}^*(\alpha))'$ .*

*Proof.* If  $A$  is an operator commuting with  $T_\alpha$ , then there exists a bundle map  $\Phi \in H_{M_n(\mathbb{C})}^\infty(\mathbb{D})$  such that  $Af(z) = \Phi(z)f(z)$  for every  $f \in L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . (Note the techniques for the Cowen-Douglas operator carry over to flat bundle case, cf [4]). And for  $f \in L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ ,  $\gamma \in \pi_1(D_0)$ , we have

$$(\Phi f)(\gamma z) = \Phi(\gamma z)f(\gamma z) = \Phi(\gamma z)\alpha(\gamma)f(z),$$

and

$$(\Phi f)(\gamma z) = \alpha(\gamma)(\Phi f)(z) = \alpha(\gamma)\Phi(z)f(z)$$

since  $(\Phi f) \in L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ . Then

$$\Phi(\gamma z) = \alpha(\gamma)\Phi(z)\alpha(\gamma)^*.$$

For every projection  $P \in (\mathcal{W}^*(T_\alpha))'$ , there a  $\Psi \in H_{M_n(\mathbb{C})}^\infty(\mathbb{D})$  such that  $Pf(z) = \Psi(z)f(z)$  for every  $f \in L_a^2(\mathbb{D}, \mathbb{C}^n, \alpha)$ .  $P^2 = P$  and  $P^* = P$  implies that  $\Psi(z)^* = \Psi(z)$  and  $\Psi(z)^2 = \Psi(z)$  for  $z \in \mathbb{D}$ , and so  $\Psi(z)$  equals a constant projection  $Q$  since  $\Psi(z)$  and  $\Psi(z)^*$  is analytic. Moreover,

$$\Psi(\gamma z) = \alpha(\gamma)\Psi(z)\alpha(\gamma)^*, \text{ for } \gamma \in \pi_1(D_0),$$

so,

$$Q = \alpha(\gamma)Q\alpha(\gamma)^*.$$

that is,  $Q \in (\mathcal{W}^*(\alpha))'$ . Conversely, it is clear that  $\Psi(z) = Q \in (\mathcal{W}^*(T_\alpha))'$  for every  $Q \in (\mathcal{W}^*(\alpha))'$ , and the correspondence is one-to-one and  $*$ .  $\square$

### 3. MULTIPLICATION OPERATOR $T_B$ ON THE BERGMAN SPACE AS A BERGMAN SHIFT

Let  $B$  be a finite Blaschke product of order  $n$ . We are ready now to show that  $T_B$  can be represented as a Bergman bundle shift. Recall the set

$$\mathcal{S} = B(\{\beta : B'(\beta) = 0\}).$$

Note that  $\mathcal{S}$  is finite and  $|\mathcal{S}| \leq n - 1$  where  $|\mathcal{S}|$  denotes the cardinality of  $\mathcal{S}$ . Let  $\mathcal{S} = \{\beta_1, \dots, \beta_k\}$  and  $l_i$  be the line obtained by joining  $\beta_i$  to boundary of  $\mathbb{D}$  with the assumption that no two lines intersect each other. Set  $\mathbb{D}_B = \mathbb{D} \setminus \cup_{i=1}^k l_i$ , clearly  $\mathbb{D}_B$  is a simply connected domain. For an open set  $U \subset \mathbb{D}$ , we define a *inverse* of  $B$  in  $U$  to be a holomorphic function  $f$  in  $U$  with  $f(U) \subset \mathbb{D}$  such that  $B(f(z)) = z$  for every  $z$  in  $U$ .

For each  $z \in \mathbb{D} \setminus \mathcal{S}$ ,  $B^{-1}(z) = \{z_1, z_2, \dots, z_n\}$  with  $z_i \neq z_j$  for  $i \neq j$  and  $B$  is one to one in some open neighborhood  $U_{z_i}$  of each point  $z_i$ . Let  $U$  be an open neighborhood of  $z$  in  $\mathbb{D}_B$  and  $\sigma_i$  be the biholomorphic map from  $U$  to  $U_{z_i}$ , for  $1 \leq i \leq n$ , such that  $B(\sigma_i(z)) = z$  for  $z \in U$ . Since  $\mathbb{D}_B$  is simply connected so by Monodromy theorem we can extend holomorphically each  $\sigma_i$  to the domain  $\mathbb{D}_B$  and we denote the extended holomorphic function by the same symbol  $\sigma_i$ .

Note that here we use inverses of  $B$ , not local inverses as in [9],[10]

**Lemma 25.** *The images of the  $\sigma_i$ 's are disjoint, that is,  $\sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B) = \emptyset$  for  $i \neq j$ .*

*Proof.* Suppose  $w_0 \in \sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B)$ , so there exist  $z_0 \in \mathbb{D}_B$  such that  $w_0 = \sigma_i(z_0) = \sigma_j(z_0)$ . Set  $\sigma = \sigma_j^{-1} \circ \sigma_i : \mathbb{D}_B \rightarrow \mathbb{D}_B$ , here  $\sigma$  is a biholomorphic map (because  $\sigma_i$ 's are biholomorphic). Since  $\mathbb{D}_B$  is simply connected so by the Riemann mapping theorem there exists a biholomorphic map  $\psi$  from  $\mathbb{D}_B$  to  $\mathbb{D}$ . Set  $f = \psi \circ \sigma \circ \psi^{-1}$ , clearly  $f$  is a biholomorphic map of  $\mathbb{D}$  and  $f(\psi(z_0)) = \psi(z_0)$ . By Schwarz lemma we have

$$f(z) = \frac{\psi(z_0) - e^{i\theta} \phi_{\psi(z_0)}(z)}{1 - e^{i\theta} \overline{\psi(z_0)} \phi_{\psi(z_0)}(z)}$$

where  $\phi_{\psi(z_0)}(z) = \frac{\psi(z_0) - z}{1 - \overline{\psi(z_0)} z}$  and  $\theta \in [0, 2\pi)$ . Thus  $\sigma = \psi^{-1} \circ f \circ \psi$  which implies that  $\sigma_i = \sigma_j \circ \psi^{-1} \circ f \circ \psi$ . Since  $\sigma_i$ 's are inverses of  $B$ , so we have  $\psi^{-1} \circ f \circ \psi(z) = z$  for all  $z \in \mathbb{D}_B$ . Thus  $\sigma_i = \sigma_j$  which is a contradiction. Hence  $\sigma_i(\mathbb{D}_B) \cap \sigma_j(\mathbb{D}_B) = \emptyset$  for  $i \neq j$ .  $\square$

**Lemma 26.** *Let  $f$  be a holomorphic function defined on a domain  $\Omega$  such that  $f(z_0) = w_0$ ,  $f^{(m-1)}(z_0) = 0$  and  $f^{(m)}(z_0) \neq 0$ . The behavior of  $f$  near  $z_0$  behave like the function  $z^m$  near 0.*

*Proof.* Since  $f(z) - w_0$  has zero of order  $m$  at  $z_0$ , we have  $f(z) = (z - z_0)^m h(z)$ , where  $h$  is an analytic function near  $z_0$  and  $h(z_0) \neq 0$ ,  $h'(z_0) \neq 0$ . Let  $g(z) = (z - z_0)h(z)^{1/m}$  then clearly  $g$  is holomorphic near  $z_0$ ,  $g'(z_0) \neq 0$  and hence  $g$  is one to one near  $z_0$ . Thus  $f(z) = g(z)^m + w_0$  is a composition of three functions, the function  $g(z)$ , followed by the map  $\zeta \mapsto \zeta^m$ , followed by translations  $\xi \mapsto \xi + w_0$ .

At point  $w$  near  $w_0$ ,  $w \neq w_0$ , has  $m$  distinct preimages  $z_1(w), \dots, z_m(w)$ . They are the  $m$  branches of  $(w - w_0)^{1/m}$  composed with  $g^{-1}(\zeta)$ . If we make branch cut and consider the principal branch  $(w - w_0)^{1/m}$  on the slit disk  $\{|w - w_0| < \delta\} \setminus (w_0 - \delta, w_0]$ , for some  $\delta > 0$ , the other branches are of the form  $e^{2\pi i j/m} (w - w_0)^{1/m}$ , and preimages of  $w$  are given by the composition  $g^{-1}(\zeta)$  and these branches,

$$z_j(w) = g^{-1}(e^{2\pi i j/m} (w - w_0)^{1/m}), \quad 1 \leq j \leq m.$$

$\square$

A detailed proof of Lemma 26 can be found in [11].

Let  $w_0 \in \mathbb{D} \setminus \mathcal{S}$  and  $U$  be an open set in  $\mathbb{D}_B$  containing  $w_0$ . Let  $\{\sigma_i\}_{i=1}^n$  be inverses of  $B$  defined on  $U$ . Let  $\gamma_i$  be a closed curve in  $\mathbb{D} \setminus \mathcal{S}$  at  $w_0$  which inclose the point  $\beta_i$  for  $1 \leq i \leq k$ . If we move  $\sigma_j$ 's along closed curve  $\gamma_i$  by analytic continuation we get a permutation on elements  $\{1, \dots, n\}$ , say  $\tau_i$ , which defines an unitary operator  $V_i$  on  $\mathbb{C}^n$  as follows:

$$V_i(x_1, \dots, x_n) = (x_{\tau_i(1)}, \dots, x_{\tau_i(n)}) \quad 1 \leq i \leq k.$$

Let  $\alpha : \pi_1(\mathbb{D} \setminus \mathcal{S}) \rightarrow \mathcal{U}(\mathbb{C}^n)$  be a representation. Since  $\pi_1(\mathbb{D} \setminus \mathcal{S})$  is a free group on  $k$  generators, the representation determined by  $k$  unitary operators  $V_1, \dots, V_k$ . The representation  $\alpha$  defines a flat unitary vector bundle  $E_B$  over  $\mathbb{D} \setminus \mathcal{S}$ . Informally, we

define a topology on  $\mathbb{D} \times \mathbb{C}^n$  so that as a point  $(z, v)$  moves continuously across the cut  $l_i$  the vector  $v$  becomes  $V_i(v)$ . Now we are ready to identify  $T_B$  as a Bergman bundle shift which is the main result of the paper. Not many arguments come from Sun, Zheng, and the first author's paper [10], especially the calculation in the following proof.

**Theorem 27.** *Let  $B$  be a finite Blaschke product of order  $n$ . Let  $T_B$  be the multiplication operator on  $L_a^2(\mathbb{D})$  by  $B$ . Let  $E_B$  be the flat unitary vector bundle over  $\mathbb{D} \setminus \mathcal{S}$  determined by  $B$ . Let  $T_{E_B}$  be the multiplication operator on  $L_a^2(E_B)$  by  $z$ . Then operator  $T_B$  is unitarily equivalent to operator  $T_{E_B}$ .*

*Proof.* Define a map  $\Gamma : L_a^2(\mathbb{D}) \rightarrow L_a^2(E_B)$  as follows

$$\Gamma(f) = \frac{1}{\sqrt{n}} [((f \circ \sigma_1)\sigma'_1, \dots, (f \circ \sigma_n)\sigma'_n)^{\text{tr}}]$$

where  $[((f \circ \sigma_1)\sigma'_1, \dots, (f \circ \sigma_n)\sigma'_n)^{\text{tr}}]$  denotes the equivalence class of  $((f \circ \sigma_1)\sigma'_1, \dots, (f \circ \sigma_n)\sigma'_n)^{\text{tr}}$ . We show that  $\Gamma$  is a unitary map:

(1)  $\Gamma$  preserves the inner product:

$$\begin{aligned} \langle \Gamma(f), \Gamma(g) \rangle &= \frac{1}{n} \left\langle [((f \circ \sigma_1)\sigma'_1, \dots, (f \circ \sigma_n)\sigma'_n)^{\text{tr}}], [((g \circ \sigma_1)\sigma'_1, \dots, (g \circ \sigma_n)\sigma'_n)^{\text{tr}}] \right\rangle \\ &= -\frac{1}{2n\pi i} \sum_{i=1}^n \int_{\sigma_i(\mathbb{D}_B)} (f \circ \sigma_i)(z) \overline{(g \circ \sigma_i)(z)} |\sigma'_i(z)|^2 dz \wedge d\bar{z} \\ &= -\frac{1}{2n\pi i} \sum_{i=1}^n \int_{\mathbb{D}_B} f(z) \overline{g(z)} dz \wedge d\bar{z} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{D}_B} f(z) \overline{g(z)} dz \wedge d\bar{z} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{D}} f(z) \overline{g(z)} dz \wedge d\bar{z} \\ &= \langle f, g \rangle. \end{aligned}$$

(2)  $\Gamma$  is onto:

Let  $[(g_1, \dots, g_n)^{\text{tr}}]$  be an element of  $L_a^2(E_B)$ . Define a function  $g$  on  $\mathbb{D}$  as follows

$$g(z) = \begin{cases} \sqrt{n} g_i \circ B(z) \cdot ((\sigma_i)'(B(z)))^{-1} & \text{if } z \in \sigma_i(\mathbb{D}_B), \quad 1 \leq i \leq n \\ \text{defined by equivalence relation} & \text{if } z \in B^{-1}(\cup_{j=1}^k l_j). \end{cases}$$

Since the values match up on the boundary by construction,  $g$  is a holomorphic function on  $\mathbb{D}$  and

$$\begin{aligned} \Gamma(g) &= [((g \circ \sigma_1) \cdot \sigma'_1, \dots, (g \circ \sigma_n) \cdot \sigma'_n)^{\text{tr}}] \\ &= [(g_1 \cdot ((\sigma_1)')^{-1} \sigma'_1, \dots, g_n \cdot ((\sigma_n)')^{-1} \sigma'_n)^{\text{tr}}] \\ &= [(g_1, \dots, g_n)^{\text{tr}}] \end{aligned}$$

(3)  $\Gamma$  intertwines  $T_B$  and  $T_{E_B}$ : For  $f \in L_a^2(\mathbb{D})$

$$\begin{aligned}
\Gamma \circ T_B(f) &= \Gamma(Bf) \\
&= \frac{1}{\sqrt{n}} [((B \circ \sigma_1)(f \circ \sigma_1) \cdot \sigma'_1, \dots, (B \circ \sigma_n)(f \circ \sigma_n) \cdot \sigma'_n)^{\text{tr}}] \\
&= \frac{1}{\sqrt{n}} [(z(f \circ \sigma_1) \cdot \sigma'_1, \dots, z(f \circ \sigma_n) \cdot \sigma'_n)^{\text{tr}}] \\
&= z \frac{1}{\sqrt{n}} [(f \circ \sigma_1) \cdot \sigma'_1, \dots, (f \circ \sigma_n) \cdot \sigma'_n)^{\text{tr}}] \\
&= T_{E_B} \circ \Gamma(f).
\end{aligned}$$

Thus

$$\Gamma \circ T_B = T_{E_B} \circ \Gamma.$$

□

Some arguments in the proof of this result is closely related to the paper [10].

**Corollary 28.**  *$(W^*(T_B))'$  is isomorphic the commutant algebra of unitary matrixes which define the the bundle.*

**Corollary 29.** *The restriction of  $T_B$  to a reducing subspace is a Bergman bundle shift also, which is corresponding to the reduction of the unitaries on that subbundle determined by that subspace.*

In [8], model spaces for the special case  $B = z^n$  is discussed explicitly.

As we mentioned earlier in this section, there is no way to put a canonical order on the set  $B^{-1}(w)$  for  $w \in \mathbb{D} \setminus \mathcal{S}$ . Different choice of lines will yield different orders unless  $\mathcal{S}$  has only one element. As a consequence, the representation of the covering group by permutation group is not unique.

Representing  $T_B$  as a bundle shift allows us to recover most of the results in [9],[10] except for two key ones: the fact that  $(W^*(T_B))'$  is abelian and its linear dimension. A more careful analysis of the covering group associated to the Riemann surface  $\{(z_1, z_2) : B(z_1) = B(z_2)\}$  for  $B$  will be required for that. Also the latter is a rational function in  $z_1$  and  $z_2$ , its polynomial numerator is closely related to the permutation covering group, most likely via Galois theory. We leave such analyses to a later paper.

**Acknowledgement:** The second and third authors thank Department of Mathematics, Texas A & M, University for warm hospitality. The work of D.K.K was supported partially in the form of a fellowship by Indo-US The Virtual Institute for Mathematics and Statistical Sciences (VI-MSS) and partially in the form of Stipend by Department of Mathematics, Texas A&M, University. The work of A.X was supported by China Scholarship council and Chongqing University of Technology.

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